Eur. Phys. J. B **12**, 119–122 (1999)

THE EUROPEAN PHYSICAL JOURNAL B

EDP Sciences

Phase transitions in a bistable system driven by two colored noises

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Received 10 August 1998 and Received in final form 23 April 1999

Abstract. The colored noise problem is studied from the point of view of consistent Markovian approximations through extending unified colored-noise approximation to the case of two-colored-noise driving systems. A bistable system simultaneously driven by multiplicative and additive colored noise is investigated by means of the extended unified colored-noise approximation. It is found that, for weak strength and color of the additive noise, the form of the stationary probability distribution changes from a unimodal to a bimodal structure via a three modal one as the correlation time of the multiplicative colored noise increases, showing the system undergoes a first order phase transition from a monostable to a bistable state. Numerical simulations support our results.

PACS. 05.40.-a Fluctuation phenomena, random processes, noise, and Brownian motion – 82.20.Db Statistical theories (including transition state)

1 Introduction

The study of nonlinear dynamical systems perturbed by colored noise has become an attractive subject in recent years. Since a white noise is not always an accurate approximation for a real noise, there emerged the need for more realistic modelings of physical systems. For example, experiments and numerical calculations on dye laser, the laser gyroscope, and bistable stochastic systems have confirmed the need for modeling noisy disturbances by colored noise [1]. On the other hand, the consequence of using colored noise instead of white noise is the same as shifting from Markovian to non-Markovian processes. This situation is difficult to describe in analytical form. In fact there exists presently no exact analytical theory (except for dichotomic noise). One is forced to search for approximative approaches.

Over the last years various approximative schemes have been presented for dealing with colored-noise problems. In contrast to the general approximation with a single colored noise [2–7] there exist relative few prior studies [8,9] wherein one accounts systematically for the mutual influence of multi-colored noises. However, in some situations we may face this problem. For instance, a complex field amplitude equation for the single-mode laser with several colored multiplicative noises was derived by us in which the source and nature of noises are clear [10]. San Miguel and Sancho, Dekker, and Fox discussed non-Markovian N-component stochastic processes with colored noises [8], mainly using correlation time expansion, respectively. Besides, we developed an effective Fokker-Planck equation for the similar problem [9] by virtue of projection operator technique and the ansatz of Hanggi *et al.* [3]. It must be pointed out that these approximative schemes are necessarily restricted to small noise correlation times.

To the authors' best knowledge, the approximative schemes can mainly be grouped into two types: (1) obtaining the *effective Fokker-Planck equation* on which many approaches are based [2,3,8,9], (2) obtaining the *effective Markovian process* for which the goal is to capture the essential features of the original non-Markovian problem [4–7]. One important development is the unified colored-noise approximation (UCNA) of Jung and Hanggi [4,5]. The theory of UCNA can be understood as follows. The original stochastic process is in terms of a non-Markovian stochastic differential equation in the relevant variable. This problem can be transformed into a Markovian process through introducing auxiliary variable and extending the number of equations. Thus we can use the conventional adiabatic elimination procedure to reduce the extended problem to an approximative Markovian process in the original variable space. The advantages of the UCNA are that the resulting Langevin equation is a truly effective Markovian description for the original non-Markovian process and it is a good approximation for both small and large correlation times, and for intermediate values of the correlation time it still gives useful approximation. These facts permit the calculations of static and transient properties by means of the standard techniques developed for white-noise processes [11]. Moreover, the concept

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of the UCNA is also applicable to multi-noise driving systems [5,12].

In this paper we shall analyze a general bistable system driven by multiplicative and additive colored noises through extending the UCNA to the case with two colored noise sources. In Section 2 we introduce our approximative scheme. In Section 3, by means of the extended form of the UCNA obtained in Section 2, we calculate the stationary probability distribution for the bistable kinetic model. Finally, in Section 4, the results are compared with numerical simulations, and discussions conclude the paper.

2 General theory

The stochastic process under investigation is described by a stochastic differential equation with two colored noises,

$$\dot{x} = f(x) + g_1(x)\,\xi_1(t) + g_2(x)\,\xi_2(t)\,, \tag{1}$$

where $f(x)$ and $g_i(x)$, $i = 1,2$, are in general nonlinear functions of x, and $\xi_i(t)$ represent independent Gaussian colored noises with zero mean and correlation

$$\langle \xi_i(t)\,\xi_j(s)\rangle = \delta_{ij}\frac{D_i}{\tau_i}\exp\left(-|t-s|/\tau_i\right). \tag{2}$$

Equation (1) with (2) is stochastically equivalent to a set of stochastic differential equations

$$\dot{x} = f(x) + g_1(x)\,\xi_1(t) + g_2(x)\,\xi_2(t)\,, \tag{3}$$

$$\dot{\xi}_i = -\frac{1}{\tau_i}\xi_i + \frac{1}{\tau_i}\Gamma_i(t)\,, \tag{4}$$

$$\langle \Gamma_i(t)\rangle = 0, \quad \langle \Gamma_i(t)\,\Gamma_j(t)\rangle = 2\delta_{ij}D_i\delta(t-s)\,.$$

Obviously, this is a three-dimensional Markovian process. Changing the state variable from x to y by $x \to y = \int^x [g_2(x')]^{-1}\,\mathrm{d}x'$, equation (1) is transformed into a stochastic process, which is additive in the colored noise $\xi_2(t)$,

$$\begin{aligned}
\dot{y} &= f_1(y) + h_1(y)\,\xi_1(t) + \xi_2(t)\,,\\
f_1(y) &= f[x(y)]/g_2[x(y)]\,,\\
h_1(y) &= g_1[x(y)]/g_2[x(y)]\,.
\end{aligned}$$

After introducing a auxiliary stochastic variable $w_1 = f_1(y) + \xi_2$, we get the equivalent Langevin equations

$$\begin{aligned}
\dot{y} &= w_1 + h_1(y)\,\xi_1(t)\,, \qquad (5)\\
\dot{w}_1 &= \left[f'(y) - \frac{1}{\tau_2}\right]w_1 + \frac{1}{\tau_2}f_1(y)\\
&\quad + f_1'(y)\,h_1(y)\,\xi_1(t) + \frac{1}{\tau_2}\Gamma_2(t)\,.
\end{aligned}$$

The prime denotes the differentiation with respect to y. By using time-scale arguments similar to those in reference [5], and considering $\tau_2 \to 0$ or ∞ , the variable w_1 in equation (5) is adiabatically eliminated. Then we get a one-dimensional stochastic process with one colored noise and one white noise, which reads in the original variable x as

$$\begin{aligned}
\dot{x} &= \tilde{f}(x) + \tilde{g}_1(x)\,\xi_1(t) + \tilde{g}_2(x)\,\Gamma_2(t)\,, \qquad (6)\\
\tilde{f}(x) &= f(x)/c(x,\tau_2)\,,\\
\tilde{g}_1(x) &= g_1(x)/c(x,\tau_2)\,,\\
\tilde{g}_2(x) &= g_2(x)/c(x,\tau_2)\,,
\end{aligned}$$

and

$$c(x,\tau_2) = 1 - \tau_2\left[f'(x) - \frac{g_2'(x)}{g_2(x)}f(x)\right].$$

Similarly, focusing on colored noise $\xi_1(t)$ and considering $\tau_1 \to 0$ or ∞, and applying the above adiabatic elimination procedure to equation (6), we finally obtain the one-dimensional approximative Markovian process with two white noise forces in the original variable space,

$$\dot{x} = \frac{1}{\gamma(x,\tau_1,\tau_2)}\left[f(x) + g_1(x)\,\Gamma_1(t) + g_2(x)\,\Gamma_2(t)\right], \tag{7a}$$

$$\gamma(x,\tau_1,\tau_2) = 1 - \sum_i \tau_i\left[f'(x) - \frac{g_i'(x)}{g_i(x)}f(x)\right], \tag{7b}$$

being valid for $\gamma(x,\tau_1,\tau_2) > 0$. From equation (7), we can easily get the stochastic equivalent Langevin equation (*i.e.*, leads to the same FPE) with one white-noise term [13]

$$\begin{aligned}
\dot{x} &= F(x) + G(x)\,\Gamma(t)\,, \qquad (8)\\
F(x) &= f(x)/\gamma(x,\tau_1,\tau_2)\,,\\
G(x) &= \left[D_1 g_1^2(x) + D_2 g_2^2(x)\right]^{1/2}/\gamma(x,\tau_1,\tau_2)\,,
\end{aligned}$$

in which $\Gamma(t)$ is Gaussian white noise with zero mean and $\langle \Gamma(t)\,\Gamma(s)\rangle = 2\delta(t-s)$.

Equation (7) is one of our main results. It is clear that, When correlation time τ_1 (or τ_2) vanishes, equation (8) is identical with equation (20) of reference [5].

3 Bistable kinetic model driven by two colored noises

Now we apply the general approximation developed above to the simplest as well as the most important model, a bistable kinetic system driven by multiplicative and additive colored noise simultaneously:

$$\dot{x} = ax - bx^3 + x\xi_1(t) + \xi_2(t)\,, \tag{9}$$

in which $\xi_i(t)$ are defined by expression (2). Equation (9) is a special case of equation (1), and expression (7b) reduces to

$$\gamma(x,\tau_1,\tau_2) = 1 - a\tau_2 + (2\tau_1 + 3\tau_2)\,bx^2. \tag{10}$$

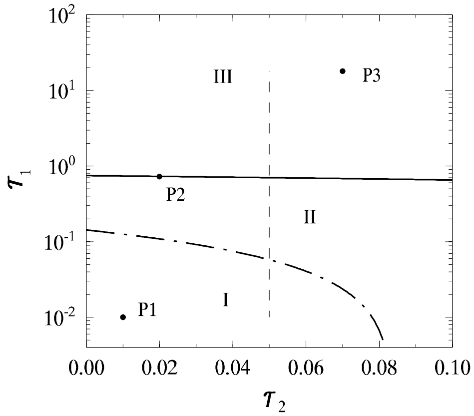

Fig. 1. Phase diagram of the system. The parameter values are $a = b = 1$, $D_1 = 1.5$, and $D_2 = 0.01$. Region I denotes the monostable phase, II the "metastable" one, and III the bistable one. The SPDs corresponding to the points P1, P2, P3 are drawn in Figure 2. The thin dashed line is the transition path taken in Figure 3.

The physical condition of validity is subject to small strength and weak color of the additive noise [5,6]. Let $R \equiv D_2/D_1$, we obtain the stationary probability distribution (SPD) of the bistable system [11]

$$P_{st}(x) = N\left|1 - a\tau_2 + (2\tau_1 + 3\tau_2)\,bx^2\right| \times \left[x^2 + R\right]^{-\left(\frac{\alpha}{2D_1}+\frac{1}{2}\right)} \exp\left[\left(\frac{1}{2}\beta x^2 - \frac{1}{4}\lambda x^4\right)/D_1\right], \quad (11)$$

where

$$\alpha = (a + bR)\left[(2\tau_1 + 3\tau_2)\,bR - (1 - a\tau_2)\right],$$
$$\beta = \left[(a + bR)(2\tau_1 + 3\tau_2) - (1 - a\tau_2)\right] b,$$
$$\lambda = (2\tau_1 + 3\tau_2)\,b^2.$$

4 Results and discussions

In general, to analyze the property of the SPD, one needs to solve the extrema. Because of the mutual influence of two colored noises, the extremal equation of SPD (11) becomes complicated which is up to 7th order indeed and is not solvable by radicals. However, we can determine the exact number of positive real roots according to the combination of Storm's theorem and Descartes' rule of signs [14]. In order to get an overview of the transition phenomena of the bistable system, the phase diagram, which represents the phase states by means of the different regions in the parameter plane (τ_1, τ_2), is depicted in Figure 1.

In the (τ_1, τ_2) parameter plane, the dash-dot curve is a spinodal line. Region I represents a monostable state. In region II, "metastable states" appear. The solid line corresponds to a first order phase transition. Region III represents a bistable state. The curves are affected by the strength of multiplicative noise D_1 as well as the one of additive noise D_2. A preliminary analysis shows that, when D_1 or D_2 varies, some regions may be broadened or narrowed, and the other region may disappear.

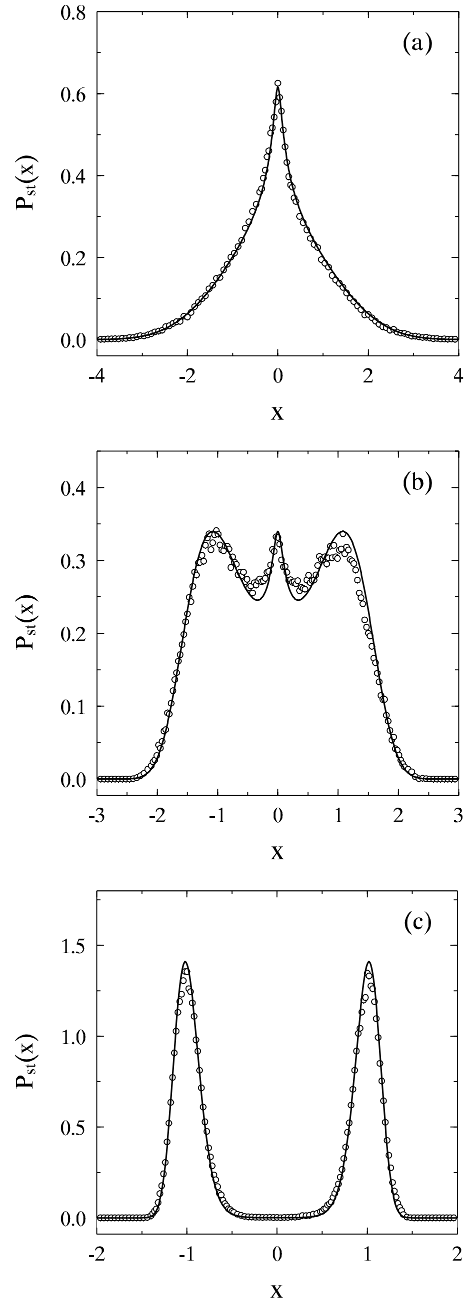

Fig. 2. The normalized SPD of equation (11) is plotted (solid line) for (a) $\tau_1 = \tau_2 = 0.01$ (monostable); (b) $\tau_1 = 0.73$, $\tau_2 = 0.02$ (transition point); (c) $\tau_1 = 18$, $\tau_2 = 0.07$ (bistable). The results of the numerical simulations are indicated by small circles.

To verify the predicted SPDs, we perform numerical simulations [15] corresponding to the original equation (9) for the three typical points P1, P2, P3 indicated in the phase diagram. Each SPD of the simulations in Figure 2 is the result of averaging 2000 independent histograms, in which one histogram is constructed through 20000 iterations in the stationary state. We find good agreement between the theory and the simulation in Figure 2. When τ_2 is fixed at a small value, we can see clearly that, in Figure 3, for small values of correlation time τ_1 in region I,

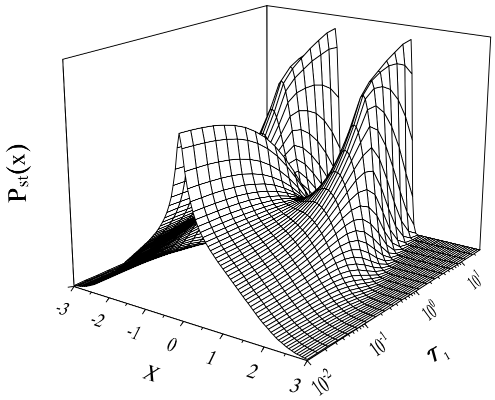

Fig. 3. Transition process through the three kinds of regions I → II → III. Plot of the unnormalized SPD as a function of the correlation time of multiplicative noise τ_1. The correlation time of the additive noise $\tau_2 = 0.05$.

the SPD has only one central peak. The system is found in monostable state. By increasing τ_1 throughout the spinodal line from region I to II, two peaks flanking the central one rise, which shows two metastable states appear. At a transition point, the two peaks become the same height as the central one. After that, we further increase τ_1, the central peak fades away and the system stays in a bistable state. These phenomena are different from the reentrance phenomena found in reference [16].

It will be of interest to mention that, because of the non-Markovian nature of the bistable system driven by multiplicative and additive colored noises, we can imagine that some novel phenomena will happen in the bistable system. Furthermore, more realistic models of physical systems need considering several colored noise sources as mentioned in the introduction, thus our extended UCNA is probably applied to some fields such as laser and chemical and biological systems.

This research has been supported by the National Natural Science Foundation of China under grant No. 19675014.

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